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Flexural–torsional buckling of open section composite columns with shear deformation

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Abstract

The paper presents the stability analysis of axially loaded, thin-walled open section, orthotropic composite columns. Vlasov's classical theory is modified to include both the transverse (flexural) shear and the restrained warping induced shear deformations. In addition to the bending stiffness matrix a (3×3) shear stiffness matrix is introduced. A closed form solution is derived for the flexural–torsional buckling load of composite columns. A simplified, approximate solution is also presented, in which the effect of the shear deformations is included using Föppl's theorem.

Keywords: Flexural–torsional; Buckling; Beam; Shear deformation; Torsion; Composite

1. Introduction

Theory of thin-walled open section beams including axial constrains (or restrained warping) was developed decades ago by Vlasov (1961), Wagner and Kappus, and today it is also discussed in university textbooks (Megson, 1990). In this classical theory it is assumed that the contour of the cross-section of the beam does not deform in its plane, the shear deformations in the middle surface of the wall are neglected, the normal stresses in the contour direction are small compared with the axial stresses. For beams, made of composite materials, the shear deformations may significantly increase the displacements, reduce the buckling load and the eigenfrequency. The shear deformation theory for transversely loaded beams was developed by Timoshenko, who also treated the effect of shear deformation on the in-plane buckling and vibration of beams (Timoshenko and Gere, 1961).

Bauld and Tzeng (1984) applied Vlasov's theory for open section composite beams with symmetrical walls neglecting the shear deformation. Bank and Bednarczyk (1988) and Barbero et al. (1993) developed simple expressions for the bending, the torsional, and the warping stiffnesses of composite beams; Massa and Barbero (1998), Bank (1990, 1987), Kobelev and Larichev (1988) included the transverse shear deformation (flexural shear strain) in the analysis. However, the effect of shear deformation of torsional

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warping was not included. Several refined theories were also proposed (see the reviews of Hodges (1990) and Friedmann (1990)), but most of them are too tedious to apply directly in the engineering practice.

To overcome these shortcomings Wu and Sun (1992) suggested a simplified theory for composite thin-walled beams, where they considered the flexural shear strains, the torsional warping induced shear strains, and the transverse shear deformations of the walls of the thin-walled beams. Their analysis, we feel, still seems rather complicated for design purposes.

Here we present an analysis in which, adopting the basic kinematic relationships of Wu and Sun, both the flexural shear strain and the torsional warping induced shear strains are considered for orthotropic beams. The analysis is applied for the flexural–torsional buckling of axially loaded composite columns, and a simple closed form solution is presented which directly shows the effect of the shear deformations. (In-plane and flexural-torsional buckling analysis of composite columns were reported by Zureick and Scott (1997), Zureick and Steffen (2000), Barbero and Raftoyiannis (1993), Barbero and Tomblin (1993) (see also the literature survey of Zureick and Scott, 1997). The effect of shear deformation on the in-plane buckling was considered but the effect of shear deformation on the torsional and flexural-torsional buckling was not taken into account. This effect will be included and discussed in the present analysis.) The vibration of composite beams will be presented in a companion paper (Kollár, 2001).

2. Problem statement

We consider prismatic beams with thin-walled open cross-sections. The walls of the beams may consist of a single layer or of several layers, each layer may be made of composite materials. The layup of the walls can be unsymmetrical, however each wall must be "orthotropic", which means that axial stresses do not cause shear strains in the wall.

The length of the column is L. The possible end conditions of the column are: (i) both ends are simply supported, (ii) both ends are fixed, (iii) one end is fixed and the other is free (Fig. 1). (At a simply supported end the transverse displacements and the rotation of the beam about the beam's axis are prevented and there are no axial constrains.) The column is subjected to concentrated compressive force ($\widehat{N}_0$) at the centroid of the cross-section at the end of the column. (The centroid is defined such that the force acting here does not cause bending of the column.) The force is increased until the column buckles, the corresponding force is called the buckling load, and is denoted by $\widehat{N}_{cr}$. We are interested in the buckling load of the column, local buckling of the wall segments are not considered, however it must be noted that for short composite columns, isolated local mode often governs the buckling (Barbero and DeVivo, 2000).

In the analysis we assume that the column behaves in a linearly elastic manner and the deformations are small. The contour of the cross-section of the column does not deform in its plane; the normal stresses in the contour direction are small compared with the axial stresses. The shear deformations in the plane of the walls are taken into account.

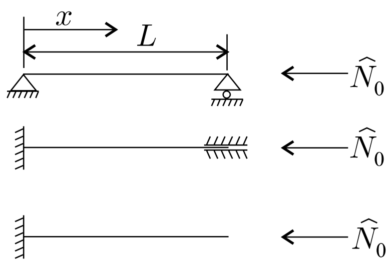


Fig. 1. End conditions for composite beams.

3. Vlasov's theory of thin-walled open section beams (no shear deformation)

In this section we summarize the basic equations of Vlasov's theory. The behavior of the isotropic beam can be characterized by the equilibrium, strain–displacement, and stress–strain (or constitutive) equations.

The *equilibrium equations* for transversely loaded beams are (Timoshenko and Gere, 1961) as follows

$$\begin{aligned} \frac{\mathrm{d}\widehat{M}_z}{\mathrm{d}x} &= \widehat{V}_y, \qquad \frac{\mathrm{d}\widehat{V}_y}{\mathrm{d}x} = -p_y \\ \frac{\mathrm{d}\widehat{M}_y}{\mathrm{d}x} &= \widehat{V}_z, \qquad \frac{\mathrm{d}\widehat{V}_z}{\mathrm{d}x} = -p_z \\ \frac{\mathrm{d}\widehat{M}_\omega}{\mathrm{d}x} &= \widehat{T}_\omega, \qquad \frac{\mathrm{d}\widehat{T}_{\mathrm{SV}}}{\mathrm{d}x} + \frac{\mathrm{d}\widehat{T}_\omega}{\mathrm{d}x} = -t \end{aligned} \tag{1}$$

where p_y and p_z are the distributed loads acting at the shear center in the y and z directions, respectively, t is the distributed moment (Fig. 2), $\widehat{V}_y$ and $\widehat{V}_z$ are the transverse shear forces (acting at the shear center), and $\widehat{M}_y$ and $\widehat{M}_z$ are the bending moments (Fig. 3). The torque, $\widehat{T}$, consists of two terms

$$\widehat{T} = \widehat{T}_{\mathrm{SV}} + \widehat{T}_\omega \tag{2}$$

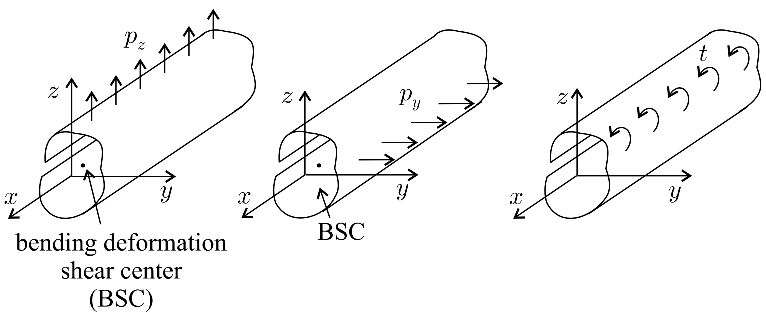


Fig. 2. Loads acting on the beam.

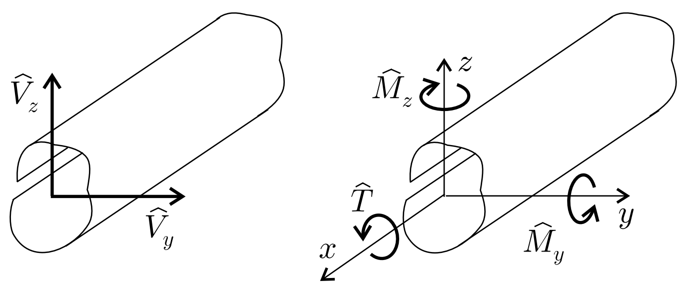


Fig. 3. Forces acting on the beam.

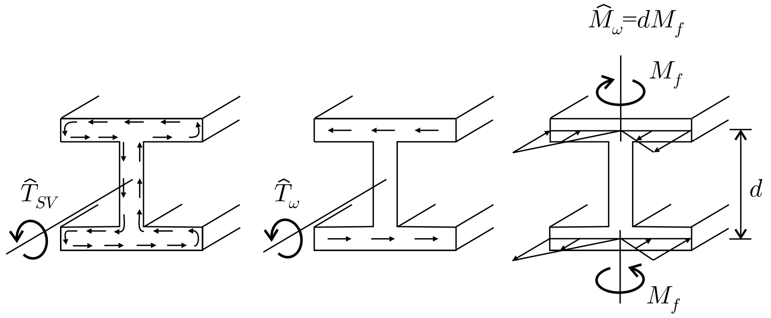


Fig. 4. Saint-Venant torque (left), warping induced torque (middle), and the bimoment (right) acting on an I-beam.

where $\widehat{T}_{\mathrm{SV}}$ is the Saint-Venant torque, which is carried by the "distributed moments" or the shear stresses as illustrated in Fig. 4, left; and $\widehat{T}_\omega$ is the warping induced torque, which is carried by the resultant shear flow as illustrated in Fig. 4, middle. $\widehat{M}_\omega$ is the bimoment or moment couple (which is illustrated for an I-beam in Fig. 4, right).

The displacements of the axis in the y and z directions are denoted by v and w, while the rotation of the cross-section (about the beam's axis) by ψ. The axial displacement $\overline{u}$ of an arbitrary point of the wall midplane, located at distances y and z from the centroid, is

$$\overline{u} = u - y\frac{\mathrm{d}v}{\mathrm{d}x} - z\frac{\mathrm{d}w}{\mathrm{d}x} - 2A_\mathrm{s}\frac{\mathrm{d}\psi}{\mathrm{d}x} \tag{3}$$

where u is the displacement of the axis attached to the centroid, and A_s is the area swept out by a generator, rotating about the center of twist, from the point of zero warping (Megson, 1990). The axial strain is the first derivative of the axial displacement

$$\epsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} - y\frac{\mathrm{d}^2v}{\mathrm{d}x^2} - z\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - 2A_\mathrm{s}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} \tag{4}$$

This strain results in axial stresses (σ_x), the appropriate resultants of which are the bending moments and the bimoment (Megson, 1990)

$$\widehat{M}_z = \int_{(S)}\int_{(h)} y\sigma_x\,\mathrm{d}\zeta\,\mathrm{d}s, \quad \widehat{M}_y = \int_{(S)}\int_{(h)} z\sigma_x\,\mathrm{d}\zeta\,\mathrm{d}s, \quad \widehat{M}_\omega = \int_{(S)}\left(2A_\mathrm{s}\int_{(h)}\sigma_x\,\mathrm{d}\zeta\right)\mathrm{d}s \tag{5}$$

where ζ is the coordinate perpendicular to the wall, and s is a coordinate along the circumference, h is the thickness of the wall and S is the length of the circumference.

The deformation components, referred to as generalized strains, are related to the displacements by the following relationships:

$$\frac{1}{\rho_z} = -\frac{\mathrm{d}^2v}{\mathrm{d}x^2}, \quad \frac{1}{\rho_y} = -\frac{\mathrm{d}^2w}{\mathrm{d}x^2}, \quad \Gamma = -\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}, \quad \vartheta = \frac{\mathrm{d}\psi}{\mathrm{d}x} \tag{6}$$

where ρ_z and ρ_y are the radii of curvatures in the x–y and x–z planes, ϑ is the rotation of cross-section per unit length. Γ is proportional to the twist induced axial strain (Eq. (4), $\Gamma \sim \epsilon_x$), and hence, it is also proportional to the bimoment ($\Gamma \sim \widehat{M}_\omega$).

The internal forces are related to the generalized strains by the force–strain relationships

$$\begin{Bmatrix} \widehat{M}_z \\ \widehat{M}_y \\ \widehat{M}_\omega \end{Bmatrix} = \begin{bmatrix} EI_{zz} & EI_{yz} & 0 \\ EI_{yz} & EI_{yy} & 0 \\ 0 & 0 & EI_\omega \end{bmatrix} \begin{Bmatrix} \frac{1}{\rho_z} \\ \frac{1}{\rho_y} \\ \Gamma \end{Bmatrix} = \left[EI_{ij}\right] \begin{Bmatrix} \frac{1}{\rho_z} \\ \frac{1}{\rho_y} \\ \Gamma \end{Bmatrix} \tag{7}$$

$$\widehat{T}_{\mathrm{SV}} = GI_{\mathrm{t}}\vartheta \tag{8}$$

where EI_{ij} is the bending stiffness, GI_{t} is the torsional stiffness, and EI_ω is the warping stiffness.

3.1. Application for orthotropic composite beams

The equilibrium equations (Eq. (1)) and the strain–displacement relationships (Eq. (6)), developed for isotropic beams, are directly applicable to composite beams. The force–strain relationships must be modified. It was shown in Massa and Barbero (1998) that for composite beams with orthotropic walls Eqs. (7) and (8) can be applied by replacing the isotropic beam stiffnesses by the stiffnesses of composite beams.

Isotropic beams Composite beams

$$\begin{array}{rcl} EA & \Rightarrow & \widehat{EA} \\ EI_{yy}, EI_{zz}, EI_{yz} & \Rightarrow & \widehat{EI}_{yy}, \widehat{EI}_{zz}, \widehat{EI}_{yz} \\ GI_{\mathrm{t}} & \Rightarrow & \widehat{GI}_{\mathrm{t}} \\ EI_\omega & \Rightarrow & \widehat{EI}_\omega \end{array} \tag{9}$$

The expressions of the stiffnesses of composite beams are given in Table 1.

4. Vlasov's theory taking the shear deformations into account

When there is no shear deformation in the beam, the cross-section remains perpendicular to the axis of the beam, and the rotations of the cross-section in the x–y or x–z planes (χ_y and χ_z) are equal to the first derivatives of the corresponding displacements. In the x–y plane we have $\mathrm{d}v/\mathrm{d}x = \chi_y$ (Fig. 5, left). In the case of shear deformation, the angle between the cross-section and the normal of the axis is denoted by γ, and in the x–y plane we write (Fig. 5, right)

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \chi_y + \gamma_y \tag{10}$$

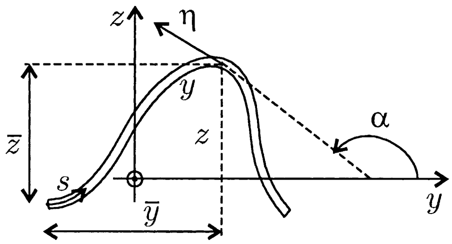
Similarly, in the x–z plane we have

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \chi_z + \gamma_z \tag{11}$$

In these equations χ_y and χ_z are the rotations of the cross-section; γ_y and γ_z are the shear strains in the x–y and x–z planes, respectively. The first terms in these equations are the bending induced deformations, while the second terms are the shear induced deformations (Fig. 5, right) which are neglected in the classical beam theory.

In the case of restrained warping shear flow arises in the wall of the beam, however, in Vlasov's theory, the shear flow induced shear strains are neglected. In reality, the twist of the beam should be obtained from the warping of the cross-section and the shear deformation of the wall. Here we observe that, similarly to the in-plane deformations of the beams (Eq. (10)), the first derivative of the twist consists of two parts (Wu and Sun, 1992)

Table 1
Coordinates of the centroid; tensile, bending, and torsional stiffnesses of an open beam with unsymmetrical orthotropic walls[a]



Tensile stiffness

$$\widehat{EA} = \int_{(S)} \frac{\widetilde{\delta}_{11}}{D}\,\mathrm{d}S$$

Coordinates of the centroid

$$y_c = \frac{\int_{(S)} \left(\bar{y}_\kappa \frac{\widetilde{\delta}_{11}}{D} + \frac{\widetilde{\beta}_{11}}{D}\sin\alpha\right)\mathrm{d}S}{\int_{(S)} \frac{\widetilde{\delta}_{11}}{D}\,\mathrm{d}S} \qquad z_c = \frac{\int_{(S)} \left(\bar{z}_\kappa \frac{\widetilde{\delta}_{11}}{D} - \frac{\widetilde{\beta}_{11}}{D}\cos\alpha\right)\mathrm{d}S}{\int_{(S)} \frac{\widetilde{\delta}_{11}}{D}\,\mathrm{d}S}$$

Bending stiffnesses

$$\widehat{EI}_{zz} = \int_{(S)} \left[\frac{\widetilde{\delta}_{11}}{D} y^2 - \frac{2\widetilde{\beta}_{11}}{D} y \sin\alpha + \frac{\widetilde{\alpha}_{11}}{D}\sin^2\alpha\right]\mathrm{d}S$$

$$\widehat{EI}_{yy} = \int_{(S)} \left[\frac{\widetilde{\delta}_{11}}{D} z^2 + \frac{2\widetilde{\beta}_{11}}{D} z \cos\alpha_\kappa + \frac{\widetilde{\alpha}_{11}}{D}\cos^2\alpha\right]\mathrm{d}S$$

$$\widehat{EI}_{yz} = \int_{(S)} \left[\frac{\widetilde{\delta}_{11}}{(D)} yz - \frac{\widetilde{\beta}_{11}}{D}(z\sin\alpha - y\cos\alpha) - \frac{\widetilde{\alpha}_{11}}{D}\cos\alpha\sin\alpha\right]\mathrm{d}S$$

Torsional stiffnesses

$$\widehat{GI}_{\mathrm{t}} = 4\int_{(S)} \frac{1}{\widetilde{\delta}_{66}}\,\mathrm{d}s$$

$\widehat{EI}_\omega$ and the location of the shear center are obtained from the expressions for isotropic beams by replacing Eh by $\widetilde{\delta}_{11}/D$.
where $(D) = (\widetilde{\alpha}_{11})(\widetilde{\delta}_{11}) - (\widetilde{\beta}_{11})^2$, and $\widetilde{\alpha}_{ij}$, $\widetilde{\beta}_{ij}$, $\widetilde{\delta}_{ij}$ are the elements of the compliance matrix of the wall

$$\begin{bmatrix} \widetilde{\alpha}_{ij} & \widetilde{\beta}_{ij} \\ \widetilde{\beta}_{ij} & \widetilde{\delta}_{ij} \end{bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix}^{-1} \quad i,j = 1,2,6$$

[a] For symmetrical walls $\widetilde{\beta}_{11} = 0$. For single layer walls $\widetilde{\delta}_{11}/D$, $\widetilde{\alpha}_{11}/D$, and $1/\widetilde{\delta}_{66}$ must be replaced by Eh, $Eh^3/12$, and Gh, respectively.

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \vartheta_{\mathrm{B}} + \vartheta_{\mathrm{S}} \tag{12}$$

The first term corresponds to the case when there is no shear deformation and the cross-section warps (Fig. 6, left), while the second term for the case when there is only shear deformation (Fig. 6, right) and there is no warping.

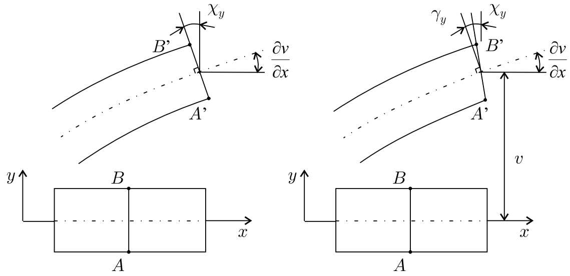


Fig. 5. Illustration of the deformations of the beam in the x–y plane without shear deformations (left), and with shear deformations (right).

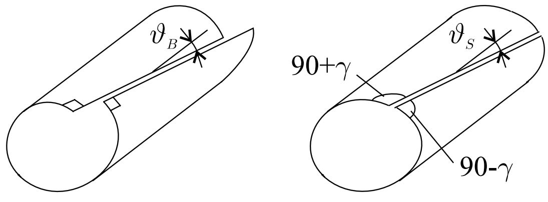


Fig. 6. Twist induced warping (left) and shear deformations (right).

The displacements of a beam with shear deformation are characterized by the functions v, w, and ψ, together with χ_y, χ_z, and ϑ_B. Here we define the following generalized strains

$$\begin{aligned} \kappa_z &= -\frac{d\chi_y}{dx}, \qquad \gamma_y = \frac{dv}{dx} - \chi_y \\ \kappa_y &= -\frac{d\chi_z}{dx}, \qquad \gamma_z = \frac{dw}{dx} - \chi_z \\ \Gamma &= -\frac{d\vartheta_B}{dx}, \qquad \vartheta_S = \frac{d\psi}{dx} - \vartheta_B \\ \vartheta &= \frac{d\psi}{dx} \end{aligned} \tag{13}$$

where γ_y and γ_z are the transverse shear strains, ϑ_S is the shear deformation caused by the rotation per unit length, all of which are assumed to be zero in Vlasov's theory. Note that κ_z is not equal to $1/\rho_z = -(d^2v/dx^2)$.

When the shear deformations are not neglected, the axial displacement $\overline{u}$ of an *arbitrary point* of the cross-section, located at distances y and z from the centroid, is (Wu and Sun, 1992)

$$\overline{u} = u - y\chi_y - z\chi_z - 2A_s\vartheta_B \tag{14}$$

The axial strain is the first derivative of the axial displacement

$$\epsilon_x = \frac{du}{dx} - y\frac{d\chi_y}{dx} - z\frac{d\chi_z}{dx} - 2A_s\frac{d\vartheta_B}{dx} \tag{15}$$

This strain results in axial stresses (σ_x), the appropriate resultants of which are the bending moments and the bimoment (Eq. (5)). By comparing Eqs. (15) and (4), from Eqs. (5) and (7) we obtain that the moments are related to the above defined strains by the bending stiffness matrix (Eq. (7))

$$\begin{Bmatrix} \widehat{M}_z \\ \widehat{M}_y \\ \widehat{M}_\omega \end{Bmatrix} = \begin{bmatrix} EI_{zz} & EI_{yz} & 0 \\ EI_{yz} & EI_{yy} & 0 \\ 0 & 0 & EI_\omega \end{bmatrix} \begin{Bmatrix} \kappa_z \\ \kappa_y \\ \Gamma \end{Bmatrix} = \left[EI_{ij}\right] \begin{Bmatrix} \kappa_z \\ \kappa_y \\ \Gamma \end{Bmatrix} \tag{16}$$

The Saint-Venant torque is (Eq. (8))

$$\widehat{T}_{\mathrm{SV}} = GI_{\mathrm{t}}\vartheta \tag{17}$$

The walls of the beam are orthotropic, hence the axial stresses and, consequently, the moments and the bimoment do not cause shear deformations of the walls. We write formalistically that the shear deformations are related to the shear forces and the warping induced torque through the shear stiffness matrix

$$\begin{Bmatrix} \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{S}_{yy} & \widehat{S}_{yz} & \widehat{S}_{y\omega} \\ \widehat{S}_{yz} & \widehat{S}_{zz} & \widehat{S}_{z\omega} \\ \widehat{S}_{y\omega} & \widehat{S}_{z\omega} & \widehat{S}_{\omega\omega} \end{bmatrix} \begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta_{\mathrm{S}} \end{Bmatrix} = \left[\widehat{S}_{ij}\right] \begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta_{\mathrm{S}} \end{Bmatrix} \tag{18}$$

The inverse of this equation yields

$$\begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta_{\mathrm{S}} \end{Bmatrix} = \begin{bmatrix} \widehat{s}_{yy} & \widehat{s}_{yz} & \widehat{s}_{y\omega} \\ \widehat{s}_{yz} & \widehat{s}_{zz} & \widehat{s}_{z\omega} \\ \widehat{s}_{y\omega} & \widehat{s}_{z\omega} & \widehat{s}_{\omega\omega} \end{bmatrix} \begin{Bmatrix} \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix} \tag{19}$$

where $\left[\widehat{s}_{ij}\right]$ is the shear compliance matrix, defined as

$$\begin{bmatrix} \widehat{s}_{yy} & \widehat{s}_{yz} & \widehat{s}_{y\omega} \\ \widehat{s}_{yz} & \widehat{s}_{zz} & \widehat{s}_{z\omega} \\ \widehat{s}_{y\omega} & \widehat{s}_{z\omega} & \widehat{s}_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \widehat{S}_{yy} & \widehat{S}_{yz} & \widehat{S}_{y\omega} \\ \widehat{S}_{yz} & \widehat{S}_{zz} & \widehat{S}_{z\omega} \\ \widehat{S}_{y\omega} & \widehat{S}_{z\omega} & \widehat{S}_{\omega\omega} \end{bmatrix}^{-1} \tag{20}$$

The elements of the shear compliance matrix $\left[\widehat{s}_{ij}\right]$ are determined in the Section 4.1.

4.1. Calculation of the shear stiffness

To determine the shear compliance matrix we consider an element of length ΔL of the beam. The two faces of the element are subjected to equal and opposite shear forces, $\widehat{V}_y$, $\widehat{V}_z$, and to a torque, $\widehat{T}$; and also to axial stresses (σ_x) (which are different at the two faces, Fig. 7, top). Here the Saint-Venant torque is neglected, and $\widehat{T} = \widehat{T}_\omega$. The work ($W$) done by the external forces are equal to the strain energy (U) of the beam

$$U = W \tag{21}$$

The axial stresses cause bending (Fig. 7, left), while the shear forces cause shear deformation of the beam (Fig. 7, right). The axial stress induced deformations are related to the moments and bimoment through the bending and warping stiffnesses ($EI_{yy}, EI_{zz}, EI_{yz}, EI_\omega$), and are not presented. Here, only the shear induced strains and displacements are considered. For a thin-walled beam the shear stresses can be represented by the shear flow (q) in the wall

$$q = \widehat{V}_y q_y + \widehat{V}_z q_z + \widehat{T}_\omega q_\omega \tag{22}$$

where q_y, q_z, and q_ω are the shear flows caused by unit shear loads $\widehat{V}_y = 1$, $\widehat{V}_z = 1$ and a unit torque $\widehat{T}_\omega = 1$, respectively. These shear flows can be calculated according to the classical analysis of thin-walled beams (Megson, 1990). (Hence we apply the classical assumptions of the "first order shear theory", i.e. we calculate the shear flow, q by neglecting the shear deformations, and we determine the shear deformation from this shear flow.)

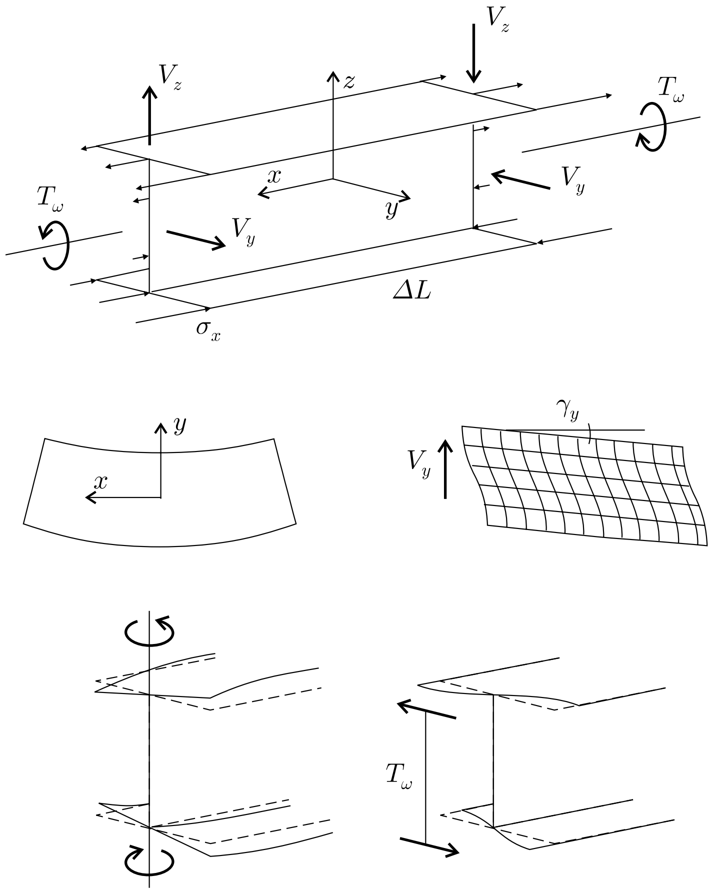


Fig. 7. Deformations caused by axial stresses (left) and shear force and torque (right).

The strain energy is

$$U = \frac{1}{2}\int \gamma q \,\mathrm{d}s\,\Delta L \tag{23}$$

where γ is the shear strain in the wall, which is related to the shear flow in a composite element by $\gamma = \widetilde{\alpha}_{66}q$, where $\widetilde{\alpha}_{66}$ is the 3, 3 element of the compliance matrix of the wall (Table 1). (For a single layer $\widetilde{\alpha}_{66} = 1/Gh$.) By utilizing this expression, Eq. (23) becomes

$$U = \frac{1}{2}\int \widetilde{\alpha}_{66}q^2 \,\mathrm{d}s\,\Delta L \tag{24}$$

The work done by the external forces due to shear deformations is

$$W = \underbrace{\frac{1}{2}\widehat{V}_y\gamma_y\,\Delta L}_{\substack{\text{due to the shear}\\ \text{displacement}\\ \text{in the } y \text{ direction}}} + \underbrace{\frac{1}{2}\widehat{V}_z\gamma_z\,\Delta L}_{\substack{\text{due to the shear}\\ \text{displacement}\\ \text{in the } z \text{ direction}}} + \underbrace{\frac{1}{2}\widehat{T}_\omega\vartheta_{\mathrm{S}}\,\Delta L}_{\substack{\text{due to the twist}\\ \text{induced shear}\\ \text{displacement}}} + \underbrace{\frac{1}{2}\left(\int_{\text{left face}}\sigma_x u_x\,\mathrm{d}A - \int_{\text{right face}}\sigma_x u_x\,\mathrm{d}A\right)}_{\text{due to the warping (neglected)}} \tag{25}$$

The work done by the axial stresses on the shear force induced warping is neglected.

Introducing Eqs. (24), (25), (19), and (22) into Eq. (21), and performing algebraic manipulations, we obtain

$$\begin{aligned}&\widehat{V}_y^2\frac{1}{2}\hat{s}_{yy}+\widehat{V}_z^2\frac{1}{2}\hat{s}_{zz}+\widehat{T}_\omega^2\frac{1}{2}\hat{s}_{\omega\omega}+\widehat{V}_y\widehat{V}_z\hat{s}_{yz}+\widehat{V}_y\widehat{T}_\omega\hat{s}_{y\omega}+\widehat{V}_z\widehat{T}_\omega\hat{s}_{z\omega}\\&\quad=\widehat{V}_y^2\frac{1}{2}\int\widetilde{\alpha}_{66}q_y^2\,\mathrm{d}s+\widehat{V}_z^2\frac{1}{2}\int\widetilde{\alpha}_{66}q_z^2\,\mathrm{d}s+\widehat{T}_\omega^2\frac{1}{2}\int\widetilde{\alpha}_{66}q_\omega^2\,\mathrm{d}s+\widehat{V}_y\widehat{V}_z\int\widetilde{\alpha}_{66}q_yq_z\,\mathrm{d}s+\widehat{V}_y\widehat{T}_\omega\\&\qquad\times\int\widetilde{\alpha}_{66}q_yq_\omega\,\mathrm{d}s+\widehat{V}_z\widehat{T}_\omega\int\widetilde{\alpha}_{66}q_zq_\omega\,\mathrm{d}s\end{aligned}\tag{26}$$

This equation must be satisfied for arbitrarily chosen values of $\widehat{V}_y$, $\widehat{V}_z$, and $\widehat{T}_\omega$, hence we have

$$\begin{aligned}&\widehat{s}_{yy}=\int\widetilde{\alpha}_{66}q_y^2\,\mathrm{d}s,\quad\widehat{s}_{zz}=\int\widetilde{\alpha}_{66}q_z^2\,\mathrm{d}s,\quad\widehat{s}_{\omega\omega}=\int\widetilde{\alpha}_{66}q_\omega^2\,\mathrm{d}s\\&\widehat{s}_{z\omega}=\int\widetilde{\alpha}_{66}q_zq_\omega\,\mathrm{d}s,\quad\widehat{s}_{y\omega}=\int\widetilde{\alpha}_{66}q_yq_\omega\,\mathrm{d}s\quad\widehat{s}_{yz}=\int\widetilde{\alpha}_{66}q_zq_y\,\mathrm{d}s\end{aligned}\tag{27}$$

These expressions of the shear compliances were evaluated for selected thin-walled composite beams and are presented in the appendix of (Kollár, 2001).

4.2. Shear center, principal directions

The axial stresses induced deformations and displacements are included in Vlasov's theory (Megson, 1990). The shear deformations cause further displacements. In the following we consider only these shear deformations of the beam. Eq. (19) shows that when the load and, consequently, the shear forces are applied at the shear center (which will be referred to as the "bending deformation shear center"), the beam will twist.

We state that there is a special location in the cross-section, referred to as the "shear deformation shear center", with the following characteristic: when the load and, consequently, the shear forces are applied at the "shear deformation shear center", there is no shear deformation induced twist. The distances of the "shear deformation shear center" from the "bending deformation shear center" in the y and z directions are given by y_{ssc} and z_{ssc} (Fig. 8).

To determine the location of the "shear deformation shear center" we place a shear force $\widehat{V}_y$ at the "shear deformation shear center", which produces a torque $\widehat{T}_\omega=-z_{ssc}\widehat{V}_y$ about the "bending deformation shear center". The shear force results in no twist, hence we can write

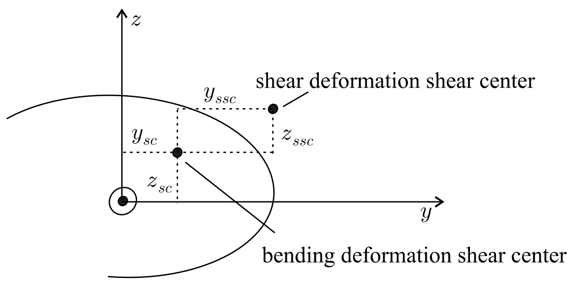


Fig. 8. The bending deformation shear center and the shear deformation shear center.

$$\vartheta_S = 0 = \widehat{s}_{y\omega}\widehat{V}_y + \widehat{s}_{\omega\omega}\widehat{T}_\omega = \widehat{s}_{y\omega}\widehat{V}_y - \widehat{s}_{\omega\omega} z_{ssc}\widehat{V}_y \tag{28}$$

which yields the coordinate of the "shear deformation shear center"

$$z_{ssc} = \frac{\widehat{s}_{y\omega}}{\widehat{s}_{\omega\omega}} \tag{29}$$

By similar reasoning, we obtain

$$y_{ssc} = -\frac{\widehat{s}_{z\omega}}{\widehat{s}_{\omega\omega}} \tag{30}$$

As a rule, y_{ssc} and z_{ssc} are not zero, hence for a beam having both shear and bending deformations we can not define a unique shear center, i.e. when a transverse load acts at the beam the beam will twist; it does not matter, at which location the beam is loaded. For example, let us consider a cantilever beam subjected to a concentrated force at the tip between the bending and shear deformation shear centers such that the tip of the beam will not rotate about the beam's axis. The shear induced rotation is a linear function of x, while the bending induced rotation is a third order function of x, consequently, all the other cross-sections of the beam will rotate about the beam's axis.

When the beam is loaded by a shear force $\widehat{V}_y$, the beam will deform in both the y–x and the z–x planes (Eq. (19)). However, there is a special direction, referred to as the principal direction, with the following characteristic: when the beam is loaded in the principal direction, shear strains occur in the plane of the shear load, but do not develop perpendicularly to this plane. The angle between the y axis and the principal direction is denoted by β. To determine β we place a unit shear force in the β direction, hence the shear forces are

$$\widehat{V}_y = \cos\beta, \qquad \widehat{V}_z = \sin\beta \tag{31}$$

while the shear strains are (Eq. (19))

$$\begin{Bmatrix} \gamma_y \\ \gamma_z \end{Bmatrix} = \begin{bmatrix} \widehat{s}_{yy} & \widehat{s}_{yz} \\ \widehat{s}_{yz} & \widehat{s}_{zz} \end{bmatrix} \begin{Bmatrix} \cos\beta \\ \sin\beta \end{Bmatrix} \tag{32}$$

The shear deformation perpendicular to the β direction is zero, hence we can write

$$\gamma = \gamma_y \sin\beta - \gamma_z \cos\beta = \left(\widehat{s}_{yy} - \widehat{s}_{zz}\right)\cos\beta\sin\beta + \widehat{s}_{yz}\left(\sin^2\beta - \cos^2\beta\right) = 0 \tag{33}$$

This equation yields the following condition for the principal direction

$$\tan 2\beta = \frac{2\widehat{s}_{yz}}{\widehat{s}_{yy} - \widehat{s}_{zz}} \tag{34}$$

As a rule, the principal direction for the shear stiffnesses differs from the principal direction for the bending stiffnesses. When the cross-section has at least one axis of symmetry, the axis of symmetry (and the coordinate perpendicular to it) are principal directions for both shear and bending deformations. For example, if the y axis is a symmetry axis of the cross-section, $EI_{yz} = \widehat{s}_{yz} = 0$.

5. Flexural–torsional buckling

The equilibrium equations of a buckled column subjected to a concentrated force at the centroid can be obtained by expressing the loads in Eq. (1) as follows (Timoshenko and Gere, 1961)

$$p_y = \widehat{N}_0\left(\frac{d^2v}{dx^2} + z_{sc}\frac{d^2\psi}{dx^2}\right)$$
$$p_z = \widehat{N}_0\left(\frac{d^2w}{dx^2} - y_{sc}\frac{d^2\psi}{dx^2}\right) \tag{35}$$
$$t = \widehat{N}_0\left(z_{sc}\frac{d^2v}{dx^2} - y_{sc}\frac{d^2w}{dx^2} + i_\omega^2\frac{d^2\psi}{dx^2}\right)$$

In these equations y_{sc} and z_{sc} are the coordinates of the "bending deformation shear center", and i_ω is the polar radius of inertia of the cross-section about the "bending deformation shear center" (Timoshenko and Gere, 1961)

$$i_\omega^2 = z_{sc}^2 + y_{sc}^2 + \frac{I_{zz} + I_{yy}}{A} \tag{36}$$

For a composite column i_ω is defined as

$$i_\omega^2 = z_{sc}^2 + y_{sc}^2 + \frac{\widehat{EI}_{zz} + \widehat{EI}_{yy}}{\widehat{EA}} \tag{37}$$

where $\widehat{EA}$, $\widehat{EI}_{zz}$, $\widehat{EI}_{yy}$ are given in Table 1.

To determine the buckling load of the column the equilibrium equations (Eqs. (1) and (35)), the strain–displacement relationships (Eq. (13)) and the constitutive equations (Eqs. (16)–(18)) must be solved taking the appropriate boundary conditions into account.

5.1. Simply supported columns

For a simply supported column, when the rotation of the cross-section of the column about the column's axis is prevented, the displacements are zero at both ends

$$v = 0, \quad w = 0, \quad \psi = 0, \quad x = 0, L \tag{38}$$

There are no axial constraints, consequently the moments are zero

$$\widehat{M}_z = 0, \quad \widehat{M}_y = 0, \quad \widehat{M}_\omega = 0, \quad x = 0, L$$

The moments are related to χ_y, χ_z, and ϑ_B (Eq. (16)), hence we can write

$$\frac{d\chi_y}{dx} = 0, \quad \frac{d\chi_z}{dx} = 0, \quad \frac{d\vartheta_B}{dx} = 0, \quad x = 0, L \tag{39}$$

We assume the displacements in the following form

$$\begin{aligned} v &= v_0 \sin\alpha x, \quad & \chi_y &= \chi_{y0}\cos\alpha x \\ w &= w_0 \sin\alpha x, \quad & \chi_z &= \chi_{z0}\cos\alpha x \\ \psi &= \psi_0 \sin\alpha x, \quad & \vartheta_B &= \vartheta_{B0}\cos\alpha x \end{aligned} \tag{40}$$

where

$$\alpha = \frac{\pi}{l} \tag{41}$$

and $v_0, \ldots, \vartheta_{B0}$ are yet unknown constants. l is the "buckling length"

$$l = \frac{L}{k} \quad k = 1, 2, \ldots \tag{42}$$

The lowest buckling force belongs to $k = 1$, hence to $l = L$.

These displacements satisfy the boundary conditions, and as it is shown below, also satisfy the differential equation system.

By introducing the displacements (Eq. (40)) into the strain–displacement relationship (Eq. (13)), the strains into the constitutive equations (Eqs. (16)–(18)), and the forces into the equilibrium equations (Eqs. (1) and (35)), we obtain from the left three equations of Eq. (1)

$$0 = \alpha[S_{ij}]\begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix}\cos\alpha x - \left([S_{ij}] + \alpha^2[EI_{ij}]\right)\begin{Bmatrix} \chi_{y0} \\ \chi_{z0} \\ \vartheta_{B0} \end{Bmatrix}\cos\alpha x \tag{43}$$

while from the right three equations of Eq. (1)

$$0 = \alpha[S_{ij}]\begin{Bmatrix} \chi_{y0} \\ \chi_{z0} \\ \vartheta_{B0} \end{Bmatrix}\sin\alpha x + \left(-\alpha^2[S_{ij}] - \alpha^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_t \end{bmatrix} + \alpha^2\widehat{N}_0[G]\right)\begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix}\sin\alpha x \tag{44}$$

In these equations $[S_{ij}]$ and $[EI_{ij}]$ are the shear and the bending stiffness matrices (see Eqs. (16) and (18)). $[G]$ is given by

$$[G] = \begin{bmatrix} 1 & 0 & z_{sc} \\ 0 & 1 & -y_{sc} \\ z_{sc} & -y_{sc} & i_\omega^2 \end{bmatrix} \tag{45}$$

Eliminating χ_{y0}, χ_{z0}, ϑ_{B0} from Eqs. (43) and (44), we obtain the following equation

$$0 = \left([S_{ij}] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_t \end{bmatrix} - [S_{ij}]\left([S_{ij}] + \alpha^2[EI_{ij}]\right)^{-1}[S_{ij}] - \widehat{N}_0[G]\right)\begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} \tag{46}$$

This equation can be rearranged (Potzta, 2000) to yield

$$0 = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_t \end{bmatrix} + \left([S_{ij}]^{-1} + \frac{1}{\alpha^2}[EI_{ij}]^{-1}\right)^{-1} - \widehat{N}_0[G]\right)\begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} \tag{47}$$

The non-trivial solutions of this equation gives three eigenvalues $\widehat{N}_{0i}$, which are identical to the buckling loads of the column $\widehat{N}_{0i} = \widehat{N}_{cr\,i}$ $(i = 1, 2, 3)$. As a rule, all of them belong to coupled flexural–torsional buckling modes of the column (Timoshenko and Gere, 1961).

When the cross-section has one plane of symmetry, one of the buckling loads belongs to a flexural buckling mode and the other two buckling loads to flexural–torsional buckling modes; while when the cross-section has two planes of symmetry, the three buckling loads belong respectively to the flexural buckling modes in the two planes of symmetry and to the pure torsional buckling (when the axis of the beam does not bend).

5.1.1. Columns with doubly symmetrical cross-sections

We consider columns in which the cross-sections are symmetrical with respect to both the y and the z axes. For such columns the bending and shear deformation shear centers are at the centroid and the principal directions for both the bending and the shear stiffnesses coincide with the y and z axes. Consequently, the bending and shear stiffness matrices simplify to

$$[EI_{ij}] = \begin{bmatrix} EI_{zz} & 0 & 0 \\ 0 & EI_{yy} & 0 \\ 0 & 0 & EI_\omega \end{bmatrix}, \qquad [S_{ij}] = \begin{bmatrix} \widehat{S}_{yy} & 0 & 0 \\ 0 & \widehat{S}_{zz} & 0 \\ 0 & 0 & \widehat{S}_{\omega\omega} \end{bmatrix} \tag{48}$$

The coordinates of the "bending deformation shear center" are zero ($y_{sc} = z_{sc} = 0$), hence matrix $[G]$ is a diagonal matrix, and Eq. (47) simplifies to

$$0 = \left(\begin{bmatrix} \frac{1}{\frac{1}{\frac{\pi^2 EI_{zz}}{l^2}} + \frac{1}{\widehat{S}_{yy}}} & 0 & 0 \\ 0 & \frac{1}{\frac{1}{\frac{\pi^2 EI_{yy}}{l^2}} + \frac{1}{\widehat{S}_{zz}}} & 0 \\ 0 & 0 & \left(\frac{1}{\frac{1}{\frac{\pi^2 EI_\omega}{l^2}} + \frac{1}{\widehat{S}_{\omega\omega}}} + GI_t \right) \frac{1}{i_\omega^2} \end{bmatrix} - \widehat{N}_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} \tag{49}$$

This equation results in three buckling forces $\widehat{N}_{cr\,z}$, $\widehat{N}_{cr\,y}$, and $\widehat{N}_{cr\,\psi}$, which correspond to the buckling about the z axis, to the buckling about the y axis, and to the pure torsional buckling (when the axis of the column twists but does not bend). By introducing the following definitions

$$\widehat{N}_{cr\,\psi} = \widehat{N}_{cr\,\omega} + \frac{1}{i_\omega^2} GI_t \tag{50}$$

$$\widehat{N}^{B}_{cr\,z} = \frac{\pi^2 EI_{zz}}{l^2}, \quad \widehat{N}^{B}_{cr\,y} = \frac{\pi^2 EI_{yy}}{l^2}, \quad \widehat{N}^{B}_{cr\,\omega} = \frac{1}{i_\omega^2} \frac{\pi^2 EI_\omega}{l^2} \tag{51}$$

$\widehat{N}_{cr\,z}$, $\widehat{N}_{cr\,y}$, $\widehat{N}_{cr\,w}$ can be calculated from Eq. (49) as follows

$$\frac{1}{\widehat{N}_{cr\,z}} = \frac{1}{\widehat{N}^{B}_{cr\,z}} + \frac{1}{\widehat{S}_{yy}}, \quad \frac{1}{\widehat{N}_{cr\,y}} = \frac{1}{\widehat{N}^{B}_{cr\,y}} + \frac{1}{\widehat{S}_{zz}}, \quad \frac{1}{\widehat{N}_{cr\,\omega}} = \frac{1}{\widehat{N}^{B}_{cr\,\omega}} + \frac{1}{\frac{1}{i_\omega^2}\widehat{S}_{\omega\omega}} \tag{52}$$

Superscript B refers to the bending deformations. The first two expressions are identical to the formulas of in-plane buckling loads of composite columns with shear deformation (Zureick and Steffen, 2000).

5.1.2. Cross-sections where the bending and shear centers coincide, and the bending and shear principal directions are identical

We take the coordinate axis in such a way that the principal directions for the bending stiffnesses are in the y and z directions. Consequently, the bending stiffness EI_{yz} is zero ($EI_{yz} = 0$). In addition we assume that the bending and shear centers coincide, and the bending and shear principal directions are identical. Consequently, $\widehat{S}_{yz}$, $\widehat{S}_{y\omega}$, $\widehat{S}_{z\omega}$ are zero, and the bending and the shear stiffness matrices simplify to Eq. (48). Correspondingly, Eq. (47) simplifies to

$$\begin{bmatrix} \widehat{N}_{cr} - \widehat{N}_{cr\,z} & 0 & \widehat{N}_{cr} z_{sc} \\ 0 & \widehat{N}_{cr} - \widehat{N}_{cr\,y} & -\widehat{N}_{cr} y_{sc} \\ \widehat{N}_{cr} z_{sc} & -\widehat{N}_{cr} y_{sc} & \left(\widehat{N}_{cr} - \widehat{N}_{cr\,\omega} - \frac{1}{i_\omega^2} GI_t \right) i_\omega^2 \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} = 0 \tag{53}$$

where $\widehat{N}_{cr\,z}$, $\widehat{N}_{cr\,y}$, $\widehat{N}_{cr\,\omega}$ are given by Eq. (52).

When the shear deformations are neglected the shear stiffnesses have to be infinite, and Eq. (53) simplifies to

$$\begin{bmatrix} \widehat{N}_{\rm cr} - \widehat{N}^{\rm B}_{{\rm cr}\,z} & 0 & \widehat{N}_{\rm cr} z_{\rm sc} \\ 0 & \widehat{N}_{\rm cr} - \widehat{N}^{\rm B}_{{\rm cr}\,y} & -\widehat{N}_{\rm cr} y_{\rm sc} \\ \widehat{N}_{\rm cr} z_{\rm sc} & -\widehat{N}_{\rm cr} y_{\rm sc} & \left(\widehat{N}_{\rm cr} - \widehat{N}^{\rm B}_{{\rm cr}\,\omega} - \frac{1}{i^2_\omega} GI_{\rm t}\right) i^2_\omega \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ \psi_0 \end{Bmatrix} = 0 \tag{54}$$

which is identical to the equation of Timoshenko and Gere (1961, p. 233).

5.2. Columns builtin at both ends

For a column builtin at both ends the boundary conditions are

$$v = 0, \quad w = 0, \quad \psi = 0, \quad x = 0, L \tag{55}$$

$$\chi_y = 0, \quad \chi_z = 0, \quad \vartheta_{\rm B} = 0, \quad x = 0, L \tag{56}$$

We assume the displacements in the following form

$$\begin{aligned} v &= v_0(1 - \cos\alpha x), & \chi_y &= -\chi_{y0}\sin\alpha x \\ w &= w_0(1 - \cos\alpha x), & \chi_z &= -\chi_{z0}\sin\alpha x \\ \psi &= \psi_0(1 - \cos\alpha x), & \vartheta_{\rm B} &= -\vartheta_{\rm B0}\sin\alpha x \end{aligned} \tag{57}$$

where $\alpha = \pi/l$ (Eq. (41)) and $v_0, \ldots, \vartheta_{\rm B0}$ are yet unknown constants. l is the "buckling length", which, for the lowest critical load, is equal to the half length of the column

$$l = \frac{L}{2} \tag{58}$$

These displacements satisfy the boundary conditions. By following the same steps as in Section 5.1, we arrive at Eq. (47). The only difference is the definition of the buckling length.

5.3. Cantilevers

For a cantilever of the length L, the buckling load can be obtained in a similar manner. Following similar steps as in Section 5.1, we arrive at Eq. (47) with the only difference that the buckling length is defined by

$$l = 2L \tag{59}$$

The steps of this analysis is not presented here.

5.4. Approximate solution

We derived a condition to calculate the natural frequencies of composite beams with arbitrary cross-sections (Eq. (47)) taking the shear deformations into account. In this section an approximate solution is presented.

We take the coordinate axis in such a way that the principal directions for the bending stiffnesses coincide with the y and z directions. Consequently, the bending stiffness EI_{yz} is zero ($EI_{yz} = 0$). In the shear stiffness matrix we neglect the elements out of the main diagonal

$$\widehat{S}_{yz} \approx \widehat{S}_{y\omega} \approx \widehat{S}_{z\omega} \approx 0 \tag{60}$$

By this approximation Eq. (47) simplifies and the buckling load can be calculated from Eq. (53). Note that formalistically Eq. (53) is identical to the well-known equation for calculating the buckling force of columns without shear deformation (Eq. (54)) when the following substitution is made

Columns without shear deformation	Columns with shear deformation

$$\begin{aligned} \widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,z} \quad &\Rightarrow \quad \left(\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,z}} + \frac{1}{\widehat{S}_{yy}} \right)^{-1} \\ \widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,y} \quad &\Rightarrow \quad \left(\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,y}} + \frac{1}{\widehat{S}_{zz}} \right)^{-1} \\ \widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,\omega} \quad &\Rightarrow \quad \left(\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,\omega}} + \frac{1}{\frac{1}{i_\omega^2}\widehat{S}_{\omega\omega}} \right)^{-1} \end{aligned} \tag{61}$$

Here $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,z}$, $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,y}$, and $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,\omega}$ correspond to buckling when the shear stiffnesses $\widehat{S}_{yy}$, $\widehat{S}_{zz}$, $\widehat{S}_{\omega\omega}$ are infinite, and GI_{t} is zero (Section 5.1.1, Eq. (51))

- $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,z} = \pi^2 EI_{zz}/l^2$ is the buckling load if the column buckles about the z axis,
- $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,y} = \pi^2 EI_{yy}/l^2$ is the buckling load if the column buckles about the y axis, and
- $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\,\omega} = \frac{1}{i_\omega^2}(\pi^2 EI_\omega/l^2)$ is the pure torsional buckling load (when the axis of the column twists but does not bend).

Superscript B indicates that only bending deformations are considered.

It can also be shown that $\widehat{S}_{yy}$, $\widehat{S}_{zz}$, $(1/i_\omega^2)\widehat{S}_{\omega\omega}$ correspond to buckling when the bending stiffnesses EI_{yy}, EI_{zz}, $EI_{\omega\omega}$ are infinite, and GI_{t} is zero

- $\widehat{S}_{yy}$ is the buckling load if the column buckles about the z axis,
- $\widehat{S}_{zz}$ is the buckling load if the column buckles about the y axis, and
- $(1/i_\omega^2)\widehat{S}_{\omega\omega}$ is the pure torsional buckling load (when the axis of the column twists but does not bend).

We observe that Eq. (61) shows the same structure as the formulas suggested by Föppl (Tarnai (1999)) to determine the buckling load of structures characterized with different stiffnesses, Föppl showed that, under certain conditions, the buckling load of structures having two stiffnessess D_1 and D_2, can be approximated as

$$N_{\mathrm{cr}} = \left(\frac{1}{N_{\mathrm{cr}1}} + \frac{1}{N_{\mathrm{cr}2}} \right)^{-1}$$

where $N_{\mathrm{cr}1}$ is the buckling load of the structure if D_2 is set equal to infinity, while $N_{\mathrm{cr}2}$ is the buckling load of the structure if D_1 is set equal to infinity.

6. Conclusion

In this paper we presented the governing equations of thin-walled open section composite columns including the effect of shear deformation both in the in-plane displacements and in the restrained warping. A closed form solution was derived for the buckling load of axially loaded columns (Eq. (47)). An approximate solution was also suggested, in which the well-known solution of columns without shear deformations (Eq. (54)) can be used by simply reducing three terms due to the shear deformations (Eq. (61)). This solution has the advantage that it shows directly the effect of shear deformation on the buckling load.

A numerical example, and the application of the above theory for free vibration of beams will be presented in a companion paper (Kollár, 2001).

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